# Solutions

## 1.5: Methods of Proof

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We introduced the basic proof structure of Section 1.2 because truth tables quickly became cumbersome and unrealistic to use. Again, we must introduce more complicated proofs because the large number of definitions and axioms leads to more complex statements which makes the methods of propositional logic unrealistic.

**Question 1.** Suppose $a$ and $b$ are odd integers. What can be said about their sum $a+b$? Explain your reasoning. $a = 2k_1+1$ $b = 2k_2+1$ for some $k_1$ and $k_2$ (integers)

$a+b = (2k_1+1)+(2k_2+1) = 2(k_1+k_2)+2 = 2(k_1+k_2+1)$ is even

**Question 2.** What can be said about the sum of two even integers?

Similarly

$a+b = 2k_1+2k_2 = 2(k_1+k_2)$ is even.

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**Direct Proofs:** The structure of a proof sequence in propositional logic is straightforward: in order to prove $A \Rightarrow C$, we prove a sequence of results.

$$A \Rightarrow B_1 \Rightarrow B_2 \Rightarrow \cdots \Rightarrow B_n \Rightarrow C$$

A **direct proof** in mathematics has the same logic, but we don't usually write such proofs as lists of statement and reasons. Instead, this linear chain of implications is couched in mathematical prose and written in paragraph form. Our first example of a direct proof already appeared in Theorem 1 of Section 1.4.

**Example 1.** Prove the following statement.

For all real numbers $x$, if $x > 1$, then $x^2 > 1$.

Section 1.2 Method (w/o predicates)

| | Statement | Reason |
|---|---|---|
| 1. | $x > 1$ | given (or supposed) |
| 2. | $x \cdot x > x \cdot 1$ | multiplication by positive numbers preserves inequalities |
| 3. | $x^2 > x > 1$ | multiplication, transitivity |

$\boxed{\text{A true proof}}$

Let $x > 1$ be a real number.
Then since multiplication by positive numbers preserves inequalities, we have

$$x \cdot x > x \cdot 1.$$

This simplies to $x^2 > x > 1$ where the last inequality follows by assumption.

**Rule of Thumb 1.** To prove a statement of the form $(\forall x)(P(x) \rightarrow Q(x))$ directly, begin your proof with a sentence of the form

Let $x$ be [an element of the domain], and suppose $P(x)$.

A direct proof is then a sequence of justified conclusions culminating in $Q(x)$.

**Definition 1.** An integer $x$ **divides** an integer $y$ if there is some integer $k$ such that $y = kx$. We write $x|y$ to denote $x$ divides $y$. Conversely, we say that $y$ **is a multiple of** $x$ whenever $x$ divides $y$.

**Axiom 1.** If $a$ and $b$ are integers, so are $a + b$ and $a \cdot b$.

**Example 2.** Prove the following.

For all integers $a, b$, and $c$, if $a|b$ and $a|c$, then $a|(b + c)$.

Let $a, b$, and $c$ be integers such that $a|b$ and $a|c$.
Then there are integers $k_1$ and $k_2$ such that $b = ak_1$ and $c = ak_2$.
Therefore $b + c = ak_1 + ak_2 = a(k_1 + k_2)$ and $a|(b+c)$, as desired.

**Example 3.** Prove the following.

For all integers $a, b$, and $c$, if $a|b$ and $b|c$, then $a|c$.

Let $a, b$, and $c$ be integers such that $a|b$ and $b|c$.
Then there exists integers $k_1$ and $k_2$ such that $b = ak_1$ and $c = bk_2$.
Therefore $c = bk_2 = a(k_1 k_2)$ and $a|c$.

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**Proof by Contraposition:** We have already seen that the contrapositive is equivalent to an implication statement; i.e. $(P(x) \rightarrow Q(x)) \iff (\neg Q(x) \rightarrow \neg P(x))$. Therefore, using predicate logic, we also have

$$(\forall x)(P(x) \rightarrow Q(x)) \iff (\forall x)(\neg Q(x) \rightarrow \neg P(x)).$$

Utilization of this fact is called a **proof by contraposition.**

**Rule of Thumb 2.** To prove a statement of the form $(\forall x)(P(x) \rightarrow Q(x))$ by contraposition, begin your proof with a sentence of the form

Let $x$ be [an element of the domain], and suppose $\neg Q(x)$.

A proof by contraposition is then a sequence of justifies conclusions culminating in $\neg P(x)$.

**Example 4.** Suppose that $x$ and $y$ are two positive real numbers such that the geometric mean $\sqrt{xy}$ is different from the arithmetic mean $\frac{x+y}{2}$. Show that $x \neq y$.

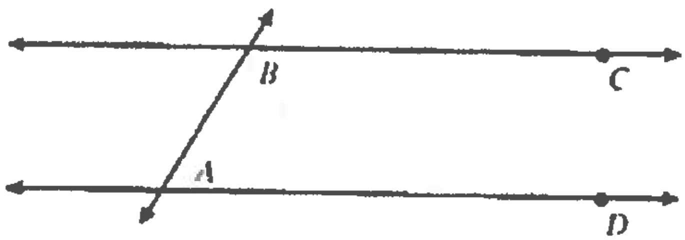
Let $x$ and $y$ be positive real numbers such that $x = y$.

Then $\sqrt{xy} = \sqrt{x^2} = x = \frac{2x}{2} = \frac{x+y}{2}$, as desired.

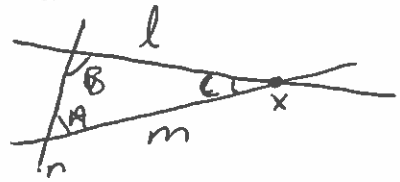
**Theorem 1.** *The sum of the measures of the angles of any triangle (in Euclidean geometry) is equal to* $180°$.

**Definition 2.** Two lines are **parallel** if they do not intersect.

**Example 5.** Prove that if two lines are cut by a transversal such that a pair of interior angles are supplementary, then the lines are parallel.




Suppose two lines $l$ and $m$ are not parallel.




Then, for any transversal $n$, the triangle $ABC$ created has sum of angles of $180°$ by Theorem 1. Since $\angle C > 0$, we must have

$$\angle A + \angle B = 180° - \angle C < 180°$$

Thus $A$ and $B$ are not supplementary. Since the argument does not depend on $n$, we are done.

**Proof by Contradiction:** Sometimes even simple-looking statements can be hard to prove directly, with or without contraposition. In this case we can try a **proof by contradiction.** Suppose we want to prove that a statement $A$ is true. Then we argue

$$\neg A \Rightarrow B_1 \Rightarrow B_2 \Rightarrow \cdots \Rightarrow B_n \Rightarrow \mathbf{F},$$

where **F** represents a statement that is always false, that is, a contradiction. By taking contrapositives we get the sequence

$$A \Leftarrow \neg B_1 \Leftarrow \neg B_2 \Leftarrow \cdots \Leftarrow \neg B_n \Leftarrow \mathbf{T},$$

where **T** is always true, that is, a tautology.

**Rule of Thumb 3.** To prove a statement $A$ by contradiction, begin your proof with the following sentence:

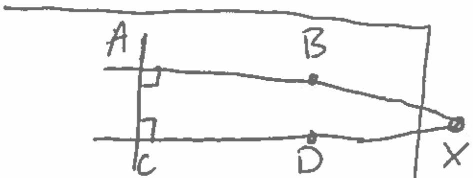
Suppose, to the contrary, that $\neg A$.

You will notice that the negations we practiced in Section 1.3 will come in handy.

Consider trying to show there are infinitely many primes.

**Example 6.** In Euclidean geometry, prove that if two lines share a common perpendicular, then the lines are parallel.

Suppose there are two lines ~~[scribble]~~ AB and CD which share a common perpendicular AC, but are not parallel. Then AB and CD must intersect at, say, X. But then $\Delta ACX$ has two right angles and hence, by Thm 1, $\angle X = 0$. This is impossible. ~~and~~ So we have a contradiction. $\Rightarrow\Leftarrow$




Somewhat counterintuitive

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**Definition 3.** An integer $n$ is **even** if $n = 2k$ for some integer $k$.

**Definition 4.** An integer $n$ is **odd** if $n = 2k + 1$ for some integer $k$.

**Axiom 2.** For all integers $n$, $\neg(n \text{ is even}) \iff (n \text{ is odd})$.

**Lemma 2.** *Let $n$ be an integer. If $n^2$ is even, then $n$ is even.*

(Contraposition)

*Proof.* Let $n$ be an integer such that $n$ is odd. Then $n = 2k_1 + 1$ for some integer $k$. Then $n^2 = (2k+1)^2 = 4k^2 + 2k + 1 = 2(2k^2 + k) + 1$ is odd. ∎

(Contradiction)

Let $n$ be an integer such that $n^2$ is even, but $n$ is odd. Then $n^2 = 2k_1$ and $n = 2k_2 + 1$ for some integers $k_1$ and $k_2$. Then

$2k_1 = n^2 = (2k_2+1)^2 = 4k_2^2 + 2k_2 + 1 = 2(2k_2^2 + k_2 + \frac{1}{2})$ and $k_1$ is not an integer. $\Rightarrow\Leftarrow$

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**Example 7.** Prove that $\sqrt{2}$ is irrational.

By way of contradiction, suppose $\sqrt{2} = \frac{a}{b}$ in lowest terms for some integers $a, b$; i.e. $\sqrt{2}$ is rational. Then $2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2$ and thus $a^2$ is even and, by Lemma 2, $a$ is even; i.e. $a = 2k$ for some integer $k$. Then $a^2 = 4k^2 = 2b^2$ and thus $b^2 = 2k^2$ is even and so $b$ is even, by Lemma 2. This contradicts that $\frac{a}{b}$ was in lowest terms. So $\sqrt{2}$ is irrational.

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**Homework. (Due Oct 8, 2018)** Section 1.5: 2, 6, 11, 14, 16

**Practice Problems.** Section 1.5: 1-9 (odd), 13, 17-20, 25